

Semiclassical Geometry of Quantum Line Bundles and Morita Equivalence of Star Products

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Abstract

In this paper we show how deformation quantization of line bundles over a Poisson manifold M produces a canonical action Φ of the Picard group $\text{Pic}(M) \cong H^2(M, \mathbb{Z})$ on the moduli space of equivalence classes of differential star products on M , $\text{Def}_{\text{diff}}(M)$. The orbits of Φ characterize Morita equivalent star products on M . We describe the semiclassical limit of Φ in terms of the characteristic classes of star products by studying the semiclassical geometry of deformed line bundles.

1 Introduction

The notion of “representation equivalence” of objects in a given category was first made precise by Morita [26] in the context of unital rings. Since then, the concept of Morita equivalence has been studied in many other settings, including groupoids, operator algebras and Poisson manifolds (see [25] for a unified approach, with original references).

Connections between noncommutative geometry and M -theory [10] have shown that Morita equivalence is related to physical duality [32]; this motivated the study of the classification of quantum tori up to Morita equivalence [31]. One can think of (the algebra of functions on) quantum tori as objects obtained from (the algebra of functions on) ordinary tori, equipped with a constant Poisson structure, by means of *strict deformation quantization* [30]; the classification problem is to determine when constant Poisson structures θ, θ' on T^n give rise to Morita equivalent quantum tori $T_\theta \sim T_{\theta'}$.

An analogous problem can be considered in the framework of formal deformation quantization [2] (see [33, 37] for surveys). In this approach to quantization, algebras of quantum

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observables are defined by formal deformations [17] of algebras of classical observables. These deformed algebra structures are called *star products*, and they often (but not always) arise as asymptotic expansions of strict quantizations (see [19, Sec. 4] and references therein). Unlike strict quantizations, star products have been shown to exist on arbitrary Poisson manifolds [23]. In this paper, we investigate the problem of which star products define Morita equivalent deformed algebras.

We show that deformation quantization of line bundles (see e.g. [7]) over a Poisson manifold (M, π) produces a canonical action

$$\Phi : \text{Pic}(M) \times \text{Def}_{\text{diff}}(M, \pi) \longrightarrow \text{Def}_{\text{diff}}(M, \pi)$$

of the Picard group of M on $\text{Def}_{\text{diff}}(M, \pi)$, the moduli space of equivalence classes of differential star products on (M, π) . The action Φ characterizes Morita equivalent star products on (M, π) : \star and \star' are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ so that the equivalence classes $[\psi^*(\star')]$ and $[\star]$ lie in the same Φ -orbit.

We use the well-known descriptions of the set $\text{Def}_{\text{diff}}(M, \pi)$ (in terms of Fedosov-Deligne's characteristic classes in the symplectic case [3, 11, 14, 27, 38], and in terms of Kontsevich's classes of formal Poisson structures [23] in the case of arbitrary Poisson manifolds) to compute the semiclassical limit of Φ explicitly. This involves the study of the semiclassical limit of line bundle deformations over Poisson manifolds. Just as the semiclassical limit of deformations of the associative algebra structure of $C^\infty(M)$ gives rise to Poisson structures on M [9, Sec. 19.4], the semiclassical limit of deformed line bundles defines a geometric object on the underlying classical line bundle: a contravariant connection [16, 34]. Contravariant connections are analogous to ordinary connections, but with cotangent vectors playing the role of tangent ones. They define a characteristic class on line bundles over Poisson manifolds, called the Poisson-Chern class. We show explicitly how the semiclassical limit of Φ “twists” characteristic classes of star products by Poisson-Chern classes.

As a result, it follows that, when M is symplectic, with free $H^2(M, \mathbb{Z})$, the action Φ is faithful, and one gets a parametrization of star products Morita equivalent to a fixed one (up to isomorphism). The discussion also provides an integrality obstruction for Morita equivalent star products in general. In the follow-up paper [5], this integrality condition is shown to be related to Dirac's quantization condition for magnetic charges. We remark that similar results hold for “strongly” Morita equivalent hermitian star products in the sense of [6, 8] (see [5]).

The paper is organized as follows. Section 2 recalls some of the necessary background: Morita equivalence, deformations of associative algebras (star products), deformations of finitely generated projective modules (and vector bundles), and Poisson fibred algebra structures. In Section 3 we define the action Φ and show how it is related to Morita equivalence. In Section 4 we study the semiclassical geometry of quantum line bundles over Poisson manifolds, and show how contravariant connections arise in this context. In Section 5 we describe the semiclassical limit of Morita equivalent star products in terms of their characteristic classes. We have included a summary of standard results on Poisson geometry in Appendix A.

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2 Preliminaries

2.1 Morita equivalence and the Picard group

For a unital ring \mathcal{R} , let ${}_{\mathcal{R}}\mathfrak{M}$ denote the category of left \mathcal{R} -modules.

Definition 2.1 *Two unital rings \mathcal{R} and \mathcal{S} are called Morita equivalent if ${}_{\mathcal{R}}\mathfrak{M}$ and ${}_{\mathcal{S}}\mathfrak{M}$ are equivalent categories.*

Example 2.2 *Let \mathcal{R} be a unital ring and $M_n(\mathcal{R})$ be the ring of $n \times n$ matrices over \mathcal{R} . Given a left \mathcal{R} -module V , we can define a left $M_n(\mathcal{R})$ -module $\mathcal{F}(V) = V^n$, with the $M_n(\mathcal{R})$ -action given by matrix operating on vectors. The functor $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{M_n(\mathcal{R})}\mathfrak{M}$ defines an equivalence of categories [24, Thm. 17.20], and \mathcal{R} and $M_n(\mathcal{R})$ are Morita equivalent.*

Isomorphic rings are Morita equivalent. Properties of a ring \mathcal{R} which are preserved under Morita Equivalence are called *Morita invariants*. Example 2.2 shows that commutativity is *not* a Morita invariant property. However, properties such as \mathcal{R} being semisimple, artinian and noetherian are Morita invariant. Morita equivalent rings have isomorphic K -theory, isomorphic lattice of ideals and isomorphic centers; hence two commutative unital rings are Morita equivalent if and only if they are isomorphic [24, Cor. 18.42].

Let \mathcal{R} and \mathcal{S} be unital rings. An $(\mathcal{S}, \mathcal{R})$ -bimodule ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ canonically defines a functor $\mathcal{F} = ({}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}} \otimes_{\mathcal{R}} \cdot) : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{S}}\mathfrak{M}$ by

$$\mathcal{F}({}_{\mathcal{R}}V) = {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V.$$

It is clear that $\mathcal{F}({}_{\mathcal{R}}V)$ has a natural \mathcal{S} -module structure determined by $s(x \otimes v) = sx \otimes v$, $x \in \mathcal{E}$, $v \in V$. If $f : {}_{\mathcal{R}}V_1 \longrightarrow {}_{\mathcal{R}}V_2$ is a morphism, then we define $\mathcal{F}(f) : {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V_1 \longrightarrow {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V_2$ by $\mathcal{F}(f)(x \otimes v) = x \otimes f(v)$, $x \in \mathcal{E}$, $v \in V_1$.

It turns out that this way of constructing functors is very general. It follows from a theorem of Eilenberg and Watts [36] that if $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{S}}\mathfrak{M}$ is an equivalence of categories, then there exists a bimodule ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ such that ${}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} \cdot \cong \mathcal{F}$. Under this identification, the composition of functors corresponds to the tensor product of bimodules.

Example 2.3 *In the case of \mathcal{R} and $M_n(\mathcal{R})$, the functor described in Example 2.2 corresponds to the bimodule ${}_{M_n(\mathcal{R})}\mathcal{R}_{\mathcal{R}}^n$.*

Corollary 2.4 *Two unital rings \mathcal{R} and \mathcal{S} are Morita equivalent if and only if there exist bimodules ${}_s\mathcal{E}_{\mathcal{R}}$ and ${}_{\mathcal{R}}\overline{\mathcal{E}}_s$ so that ${}_{\mathcal{R}}\overline{\mathcal{E}} \otimes_s \mathcal{E}_{\mathcal{R}} \cong {}_{\mathcal{R}}\mathcal{R}_{\mathcal{R}}$ and ${}_s\mathcal{E} \otimes_{\mathcal{R}} {}_{\mathcal{R}}\overline{\mathcal{E}}_s \cong {}_s\mathcal{S}_s$ (as bimodules).*

A bimodule ${}_s\mathcal{E}_{\mathcal{R}}$ establishing a Morita equivalence is called an *equivalence bimodule*. Morita's theorem provides a characterization of such bimodules.

Definition 2.5 *A right \mathcal{R} -module $\mathcal{E}_{\mathcal{R}}$ is called a *progenerator* if it is finitely generated, projective and a generator¹.*

Theorem 2.6 (Morita's theorem) *Two unital rings \mathcal{R} and \mathcal{S} are Morita equivalent if and only if there exists a progenerator right \mathcal{R} -module $\mathcal{E}_{\mathcal{R}}$ so that $\mathcal{S} \cong \text{End}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}})$. Moreover, if ${}_s\mathcal{E}_{\mathcal{R}}$ is an equivalence bimodule, then its inverse is given by ${}_{\mathcal{R}}\overline{\mathcal{E}}_s = \text{Hom}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}}, \mathcal{R})$.*

Definition 2.7 *An idempotent $P \in M_n(\mathcal{R})$ ($P^2 = P$) is called *full* if the span of elements of the form TPS , with $T, S \in M_n(\mathcal{R})$, is $M_n(\mathcal{R})$.*

A finitely generated projective \mathcal{R} -module $\mathcal{E}_{\mathcal{R}} = P\mathcal{R}^n$ ($P \in M_n(\mathcal{R})$ idempotent) is a generator if and only if P is full [24, Remark 18.10(D)]. This provides an alternative description of Morita equivalent rings.

Theorem 2.8 \mathcal{R} and \mathcal{S} are Morita equivalent if and only if there exists $n \in \mathbb{N}$ and a full idempotent $P \in M_n(\mathcal{R})$ so that $\mathcal{S} \cong PM_n(\mathcal{R})P = \text{End}_{\mathcal{R}}(P\mathcal{R}^n)$.

We note that there is a natural group associated to any unital ring \mathcal{R} .

Definition 2.9 *We define $\text{Pic}(\mathcal{R})$ as the group of equivalence classes of self-equivalence functors $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{R}}\mathfrak{M}$, with group operation given by composition; equivalently, we can view $\text{Pic}(\mathcal{R})$ as the group of isomorphism classes of $(\mathcal{R}, \mathcal{R})$ -equivalence bimodules ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$, with group operation given by tensor products (over \mathcal{R}).*

The group $\text{Pic}(\mathcal{R})$ is called the *Picard group* of \mathcal{R} .

Remark 2.10 *Note that if ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$ is an equivalence bimodule, then the center of \mathcal{R} need not act the same on the left and right of \mathcal{E} . If \mathcal{R} is commutative and \mathcal{E} is an $(\mathcal{R}, \mathcal{R})$ -equivalence bimodule, then there exists an $(\mathcal{R}, \mathcal{R})$ -equivalence bimodule \mathcal{E}' satisfying $rx = xr$, for all $r \in \mathcal{R}$ and $x \in \mathcal{E}'$, such that $\mathcal{E} \cong \mathcal{E}'$ as right \mathcal{R} -modules (pick \mathcal{E}' of the form $P_0\mathcal{R}^n$ as a right \mathcal{R} -module and consider the identification $\mathcal{R} \longrightarrow P_0M_n(\mathcal{R})P_0$, $r \mapsto rP_0$).*

If \mathcal{R} is commutative, we denote the group of isomorphism classes of $(\mathcal{R}, \mathcal{R})$ -equivalence bimodules ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$ satisfying $rx = xr$, for all $x \in \mathcal{E}$, $r \in \mathcal{R}$, by $\text{Pic}_{\mathcal{R}}(\mathcal{R})$.

¹Recall that a right \mathcal{R} -module $\mathcal{E}_{\mathcal{R}}$ is a *generator* if any other right \mathcal{R} -module can be obtained as a quotient of a direct sum of copies of $\mathcal{E}_{\mathcal{R}}$.

Example 2.11 Let $\mathcal{R} = C^\infty(M)$, where M is a smooth manifold. As a consequence of Serre-Swan's theorem [1, Chp. XIV] (here used in the smooth category, where the compactness assumption can be dropped), $\text{Pic}_{C^\infty(M)}(C^\infty(M))$ can be identified with $\text{Pic}(M)$, the group of isomorphism classes of complex line bundles over M , with group operation given by fiberwise tensor product. The Chern class map $c_1 : \text{Pic}(M) \longrightarrow H^2(M, \mathbb{Z})$ is a group isomorphism [21, Sec. 3.8], and hence $\text{Pic}_{C^\infty(M)}(C^\infty(M)) \cong H^2(M, \mathbb{Z})$.

2.2 Deformations of associative algebras: star products

Let k be a commutative, unital ring of characteristic zero, and let \mathcal{A} be an associative k -algebra; $\mathcal{A}[[\lambda]]$ denotes the space of formal power series with coefficients in \mathcal{A} .

Definition 2.12 A formal deformation of \mathcal{A} is an associative $k[[\lambda]]$ -bilinear multiplication \star on $\mathcal{A}[[\lambda]]$ of the form

$$A \star A' = \sum_{r=0}^{\infty} C_r(A, A') \lambda^r, \quad A, A' \in \mathcal{A}, \quad (2.1)$$

where the maps $C_r : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are k -bilinear, and C_0 is the original product on \mathcal{A} ².

A formal deformation of \mathcal{A} will be denoted by $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$.

Definition 2.13 We say that two formal deformations of \mathcal{A} , $\mathcal{A}_1 = (\mathcal{A}[[\lambda]], \star_1)$ and $\mathcal{A}_2 = (\mathcal{A}[[\lambda]], \star_2)$, are equivalent if there exist k -linear maps $T_r : \mathcal{A} \longrightarrow \mathcal{A}$, $r \geq 1$, so that $T = \text{id} + \sum_{r=1}^{\infty} T_r \lambda^r : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ satisfies

$$A \star_1 A' = T^{-1}(T(A) \star_2 T(A')), \quad \forall A, A' \in \mathcal{A}[[\lambda]]. \quad (2.2)$$

Such a T is called an equivalence transformation.

The set of equivalence classes of formal deformations of \mathcal{A} is denoted by $\text{Def}(\mathcal{A})$. We denote the equivalence class of a deformation \star by $[\star]$.

If \mathcal{A} is unital, then so is any formal deformation \mathcal{A} ; moreover, any formal deformation of \mathcal{A} is equivalent to one for which the unit is the same as for \mathcal{A} [18, Sec. 14].

The group of automorphisms of \mathcal{A} , $\text{Aut}(\mathcal{A})$, acts on formal deformations by $\star' = \psi^*(\star)$ if and only if $A \star' A' = \psi^{-1}(\psi(A) \star \psi(A'))$, $A, A' \in \mathcal{A}$. Since any $k[[\lambda]]$ -algebra isomorphism $S : (\mathcal{A}[[\lambda]], \star) \longrightarrow (\mathcal{A}[[\lambda]], \star')$ is of the form $S = S_0 + \sum_{r=1}^{\infty} \lambda^r S_r$, with k -linear $S_r : \mathcal{A} \longrightarrow \mathcal{A}$, and $S_0 \in \text{Aut}(\mathcal{A})$, a simple computation shows the following proposition.

Proposition 2.14 Let \star and \star' be formal deformations of \mathcal{A} . Then they are isomorphic if and only if there exists $\psi \in \text{Aut}(\mathcal{A})$ with $[\psi^*(\star')] = [\star]$.

Let \mathcal{A} be commutative and unital.

²We extend \star to $\mathcal{A}[[\lambda]]$ using λ -linearity.

Definition 2.15 A Poisson bracket on \mathcal{A} is a Lie algebra bracket $\{\cdot, \cdot\}$ satisfying the Leibniz rule

$$\{A_1, A_2 A_3\} = \{A_1, A_2\} A_3 + A_2 \{A_1, A_3\}.$$

The pair $(\mathcal{A}, \{\cdot, \cdot\})$ is called a Poisson algebra.

For a formal deformation of \mathcal{A} (2.1), a simple computation using associativity of \star shows that

$$\{A_1, A_2\} := C_1(A_1, A_2) - C_1(A_2, A_1) = \frac{1}{\lambda}(A_1 \star A_2 - A_2 \star A_1) \bmod \lambda \quad (2.3)$$

is a Poisson bracket on \mathcal{A} [9, Sect. 16]. It is simple to check that if two formal deformations are equivalent, then they determine the same Poisson bracket through (2.3).

Let $\mathcal{A} = C^\infty(M)$ be the algebra of complex-valued smooth functions on a smooth manifold M .

Definition 2.16 A formal deformation $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ of \mathcal{A} is called a star product if each C_r is a bidifferential operator.

The set of equivalence classes of star products on M is denoted by $\text{Def}_{\text{diff}}(M)$.

A Poisson structure on a manifold M is a Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$; it can be equivalently defined by a bivector field $\pi \in \Gamma^\infty(\wedge^2 TM)$ satisfying $[\pi, \pi] = 0$,³ in such a way that $\{f, g\} = \pi(df, dg)$. We call π a Poisson tensor and (M, π) a Poisson manifold. We let

$$\text{Def}_{\text{diff}}(M, \pi) := \{[\star] \in \text{Def}_{\text{diff}}(M) \mid f \star g - g \star f = \lambda \pi(df, dg) \bmod \lambda^2\}. \quad (2.4)$$

Star products on a Poisson manifold (M, π) will be assumed to be compatible with the given Poisson structure in the sense of (2.3).

If \star and \star' are star products on (M, π) , we remark that, by Proposition 2.14, they are isomorphic if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\psi^*(\star')] = [\star]$.

2.3 Deformations of projective modules and vector bundles

Let \mathcal{A} be a unital k -algebra, and let \mathcal{E} be a right module over \mathcal{A} . Let $R_0 : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{E}$ denote the right action of \mathcal{A} on \mathcal{E} , $R_0(x, A) = x \cdot A$. Let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of \mathcal{A} and suppose there exist k -bilinear maps $R_r : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{E}$, for $r \geq 1$, such that the map

$$\mathbf{R} = \sum_{r=0}^{\infty} R_r \lambda^r : \mathcal{E}[[\lambda]] \times \mathcal{A} \longrightarrow \mathcal{E}[[\lambda]] \quad (2.5)$$

makes $\mathcal{E}[[\lambda]]$ into a module over \mathcal{A} . We will write $\mathbf{R}(x, A) = x \bullet A$, for $x \in \mathcal{E}$, $A \in \mathcal{A}$.

³ $[\cdot, \cdot]$ is the Schouten bracket [35].

Definition 2.17 We call $\mathcal{E} = (\mathcal{E}[[\lambda]], \bullet)$ a deformation of the right \mathcal{A} -module \mathcal{E} corresponding to $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$. Two deformations $\mathcal{E} = (\mathcal{E}[[\lambda]], \bullet)$, $\mathcal{E}' = (\mathcal{E}[[\lambda]], \bullet')$ are equivalent if there exists an \mathcal{A} -module isomorphism $T : \mathcal{E} \rightarrow \mathcal{E}'$ of the form $T = \text{id} + \sum_{r=1}^{\infty} T_r \lambda^r$, with k -linear maps $T_r : \mathcal{E} \rightarrow \mathcal{E}$.

Suppose \mathcal{E} is finitely generated and projective over \mathcal{A} , in which case we can write $\mathcal{E} = P_0 \mathcal{A}^n$ for some idempotent $P_0 \in M_n(\mathcal{A})$ and $n \in \mathbb{N}$. Let \mathcal{A} be a formal deformation of \mathcal{A} . It is clear that $M_n(\mathcal{A})$ can be identified with $M_n(\mathcal{A})[[\lambda]]$ as a $k[[\lambda]]$ -module, and this identification naturally defines a formal deformation of $M_n(\mathcal{A})$. We recall the following well-known fact [13, 15, 18].

Lemma 2.18 Let $P_0 \in M_n(\mathcal{A})$ be an idempotent. For any formal deformation \mathcal{A} , there exists an idempotent $\mathbf{P} \in M_n(\mathcal{A})$ with $\mathbf{P} = P_0 + O(\lambda)$.

We call \mathbf{P} a deformation of P_0 with respect to \mathcal{A} . The natural right \mathcal{A} -module structure on $\mathbf{P} \star \mathcal{A}^n$ can then be transferred to $P_0 \mathcal{A}^n[[\lambda]]$ since $P_0 \mathcal{A}^n[[\lambda]] \cong \mathbf{P} \star \mathcal{A}^n$ as $k[[\lambda]]$ -modules. An explicit isomorphism is given by the map [7, Lem. 2.3]

$$J : P_0 \mathcal{A}^n[[\lambda]] \longrightarrow \mathbf{P} \star \mathcal{A}^n, \quad P_0 x \mapsto \mathbf{P} \star x, \quad (2.6)$$

$x \in \mathcal{A}^n$. By Lemma 2.18, deformations of finitely generated projective modules (f.g.p.m.) always exist (with respect to any \mathcal{A}). Moreover, they are unique up to equivalence [6, Prop. 2.6] (and hence necessarily finitely generated and projective over \mathcal{A}). We summarize these facts in the following proposition.

Proposition 2.19 Let $\mathcal{E}_{\mathcal{A}}$ be a finitely generated projective module over \mathcal{A} , and let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation. Then there exists a deformation of $\mathcal{E}_{\mathcal{A}}$ with respect to \mathcal{A} , and any two such deformations are equivalent.

A simple computation shows that fullness of idempotents is preserved under deformations:

Lemma 2.20 Let $P_0 \in M_n(\mathcal{A})$ be an idempotent and $\mathbf{P} \in M_n(\mathcal{A})$ be a deformation of P_0 . Then P_0 is full if and only if \mathbf{P} is full.

Proposition 2.21 An equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ canonically defines a bijective map $\Phi_{\mathcal{E}} : \text{Def}(\mathcal{A}) \rightarrow \text{Def}(\mathcal{B})$ so that formal deformations related by Φ are Morita equivalent.

PROOF: Let \star be a formal deformation of \mathcal{A} , and let $\mathcal{E} = (\mathcal{E}[[\lambda]], \bullet)$ be a deformation of $\mathcal{E}_{\mathcal{A}}$ with respect to \star . Then $\mathcal{B}[[\lambda]]$ and $\text{End}(\mathcal{E})$ are isomorphic as $k[[\lambda]]$ -modules. In fact, for $\mathcal{E} = P_0 \mathcal{A}^n$, P_0 full idempotent, and \mathcal{E} given by (2.6), an explicit isomorphism is (see [7, Lem. 2.3])

$$I : P_0 M_n(\mathcal{A}) P_0[[\lambda]] \longrightarrow \mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P}, \quad P_0 L P_0 \mapsto \mathbf{P} \star L \star \mathbf{P}. \quad (2.7)$$

In general, any fixed isomorphism

$$\mathcal{B}[[\lambda]] \xrightarrow{\sim} \text{End}(\mathcal{E}) \quad (2.8)$$

induces a formal deformation \star' of \mathcal{B} , and different choices of (2.8) lead to isomorphic deformations (not necessarily equivalent). Note that an isomorphism (2.8) also defines a left module structure \bullet' on $\mathcal{E}[[\lambda]]$ over $(\mathcal{B}[[\lambda]], \star')$, and we can choose (2.8) so that \bullet' is a deformation of the module structure of ${}_B\mathcal{E}$, $B \bullet' x = B \cdot x + O(\lambda)$ (this is the case for (2.7)). A simple computation shows that any two isomorphisms (2.8) yielding deformations of ${}_B\mathcal{E}$ define equivalent deformations of \mathcal{B} . Moreover, it follows from the uniqueness part of Proposition 2.19 that this equivalence class does not depend on the choice of \bullet or element in $[\star] \in \text{Def}(\mathcal{A})$. Hence this procedure defines a canonical map

$$\Phi_{\mathcal{E}} : \text{Def}(\mathcal{A}) \longrightarrow \text{Def}(\mathcal{B}), \quad (2.9)$$

which is a bijection [7, Prop. 3.3].

It follows from the construction of $\Phi_{\mathcal{E}}$ and Lemma 2.20 that formal deformations related by $\Phi_{\mathcal{E}}$ are Morita equivalent. \square

Let $\mathcal{A} = C^\infty(M)$ be the algebra of complex-valued smooth functions on a manifold M ; by Serre-Swan's theorem, f.g.p.m. over \mathcal{A} correspond to (smooth sections of) finite dimensional complex vector bundles $E \rightarrow M$.

For a smooth complex m -dimensional vector bundle $E \rightarrow M$, let $\mathcal{E} = \Gamma^\infty(E)$ be the space of smooth sections of E , regarded as a right \mathcal{A} -module, and $\mathcal{B} = \text{End}_{\mathcal{A}}(\mathcal{E}) = \Gamma^\infty(\text{End}(E))$, the algebra of smooth sections of the endomorphism bundle $\text{End}(E) \rightarrow M$.

Let \star be a star product on M .

Definition 2.22 *A deformation quantization of E with respect to \star is a deformation of $\mathcal{E} = \Gamma^\infty(E)$ in the sense of Definition 2.17 so that the corresponding R_r (as in (2.5)) are bidifferential operators.*

The explicit map in (2.6) shows that the (always existing) deformations of \mathcal{E} can be chosen with bidifferential R_r (since we are only considering differential star products); therefore deformation quantizations of vector bundles exist (with respect to any star product) and are unique up to equivalence.

If we write $\mathcal{E} = P_0 \mathcal{A}^n$, for some idempotent $P_0 \in M_n(\mathcal{A})$, we note that the deformation of \mathcal{B} induced by the explicit map (2.7) as in (2.8) is also differential. This gives rise to a canonical bijective map (Proposition 2.21)

$$\Phi_{\mathcal{E}} : \text{Def}_{\text{diff}}(M) \longrightarrow \text{Def}_{\text{diff}}(\Gamma^\infty(\text{End}(E))). \quad (2.10)$$

2.4 Poisson fibred algebras

Poisson fibred algebras [29] arise in connection with formal deformations of noncommutative algebras, providing a generalization of Poisson algebras. We will recall the definition.

Let \mathcal{B} be a unital k -algebra, not necessarily commutative, and let \mathcal{Z} be its center.

Definition 2.23 *A Poisson fibred algebra structure on $(\mathcal{B}, \mathcal{Z})$ is a bracket*

$$\{\cdot, \cdot\} : \mathcal{Z} \times \mathcal{B} \longrightarrow \mathcal{B}$$

satisfying the following conditions:

1. *The restriction of $\{\cdot, \cdot\}$ to $\mathcal{Z} \times \mathcal{Z}$ makes \mathcal{Z} into a Poisson algebra.*
2. *The following Leibniz identities hold⁴.*

$$\{Z, B_1 B_2\} = \{Z, B_1\} B_2 + B_1 \{Z, B_2\}, \quad (2.11)$$

$$\{Z_1 Z_2, B\} = Z_1 \{Z_2, B\} + Z_2 \{Z_1, B\}. \quad (2.12)$$

Suppose \mathcal{A} is a commutative, unital k -algebra, and let $P_0 \in M_n(\mathcal{A})$ be a full idempotent. Let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of \mathcal{A} . We keep the notation

$$A_1 \star A_2 = \sum_{r=0}^{\infty} C_r(A_1, A_2) \lambda^r, \quad A_1, A_2 \in \mathcal{A}.$$

Since \mathcal{A} is commutative, it inherits a Poisson algebra structure (2.3) from \star given by

$$\{A_1, A_2\} := C_1(A_1, A_2) - C_1(A_2, A_1).$$

We saw in Section 2.3 how to define a formal deformation \star' of $\mathcal{B} = P_0 M_n(\mathcal{A}) P_0$ explicitly by choosing an idempotent $\mathbf{P} \in M_n(\mathcal{A})$ deforming P_0 :

$$L_0 \star' S_0 := I^{-1}(I(L_0) \star I(S_0)) = \sum_{r=0}^{\infty} B_r(L_0, S_0) \lambda^r, \quad L_0, S_0 \in \mathcal{B},$$

where I is as in (2.7). For $M, N \in M_n(\mathcal{A})$, let (note the abuse of notation⁵)

$$\{M, N\} := C_1(M, N) - C_1(N, M), \quad (2.13)$$

where $C_1(M, N) \in M_n(\mathcal{A})$ is defined by $C_1(M, N)_{i,j} = \sum_{r=1}^n C_1(M_{i,r}, N_{r,j})$. We will compute the expression for the bracket in $P_0 M_n(\mathcal{A}) P_0 \subseteq M_n(\mathcal{A})$ given by

$$\{L, S\}' := B_1(L, S) - B_1(S, L), \quad L, S \in \mathcal{B}, \quad (2.14)$$

in terms of $\{\cdot, \cdot\}$.

⁴These identities imply that $\{Z, 1\} = \{1, B\} = 0$, for all $Z \in \mathcal{Z}$, $B \in \mathcal{B}$.

⁵The bracket in $M_n(\mathcal{A})$ defined in (2.13) induces the bracket $\{\cdot, \cdot\}$ on \mathcal{A} through the identification of \mathcal{A} with the center of $M_n(\mathcal{A})$ in the natural way; we denote both brackets by $\{\cdot, \cdot\}$ by abuse of notation.

Proposition 2.24 For all $L_0, S_0 \in P_0 M_n(\mathcal{A}) P_0$, we have $\{L_0, S_0\}' = P_0 \{L_0, S_0\} P_0$.

PROOF: Since $I : P_0 M_n(\mathcal{A}) P_0[[\lambda]] \longrightarrow \mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P}$ and $\mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P} \subseteq M_n(\mathcal{A}) = M_n(\mathcal{A})[[\lambda]]$, we can write $I = \sum_{r=0}^{\infty} I_r \lambda^r$, where

$$I_r : P_0 M_n(\mathcal{A}) P_0 \longrightarrow M_n(\mathcal{A}).$$

A simple computation shows that $I_0(L_0) = L_0$ and

$$I_1(L_0) = C_1(P_0, L_0) + P_0 C_1(L_0, P_0) + L_0 P_1 + P_1 L_0. \quad (2.15)$$

The equations $\mathbf{P} \star I(L_0) = I(L_0)$ and $I(L_0) \star \mathbf{P} = I(L_0)$ imply that

$$P_0 C_1(P_0, L_0) + P_0 P_1 L_0 = 0, \quad \text{and} \quad C_1(L_0, P_0) P_0 + L_0 P_1 P_0 = 0. \quad (2.16)$$

It follows from (2.15) and (2.16) that $P_0 I_1(L_0) P_0 = 0$, for all $L_0 \in P_0 M_n(\mathcal{A}) P_0$. Note that if $L_0 + \lambda L_1 + \dots \in P_0 M_n(\mathcal{A}) P_0[[\lambda]]$ and $I(L_0 + \lambda L_1 + \dots) = M_0 + \lambda M_1 + \dots$, then $I_1(L_0) + I_0(L_1) = I_1(L_0) + L_1 = M_1$, and hence

$$P_0(I_1(L_0) + L_1) P_0 = L_1 = P_0 M_1 P_0.$$

But $I(L_0 \star' S_0) = I(L_0) \star I(S_0) = L_0 S_0 + \lambda(C_1(L_0, S_0) + L_0 I_1(S_0) + I_1(L_0) S_0) + \dots$. Thus

$$B_1(L_0, S_0) = P_0(C_1(L_0, S_0) + L_0 I_1(S_0) + I_1(L_0) S_0) P_0 = P_0 C_1(L_0, S_0) P_0,$$

by (2.16), and the result follows. \square

Let \mathcal{Z} denote the center of $\mathcal{B} = P_0 M_n(\mathcal{A}) P_0$, $P_0 \in M_n(\mathcal{A})$ full idempotent. The triple $(\mathcal{B}, \mathcal{Z}, \{\cdot, \cdot\}')$ is a Poisson fibred algebra [29, Prop. 1.2], and, as such, the restriction of $\{\cdot, \cdot\}'$ to $\mathcal{Z} \times \mathcal{Z}$ provides \mathcal{Z} with the structure of a Poisson algebra. We can identify \mathcal{Z} and \mathcal{A} through the algebra isomorphism $\Psi : \mathcal{A} \longrightarrow \mathcal{Z}$, $A \mapsto P_0 A P_0 = A P_0$.

Theorem 2.25 The map $\Psi : (\mathcal{A}, \{\cdot, \cdot\}) \longrightarrow (\mathcal{Z}, \{\cdot, \cdot\}')$ is an isomorphism of Poisson algebras.

PROOF: The bracket $\{\cdot, \cdot\}' : \mathcal{Z} \times \mathcal{B} \longrightarrow \mathcal{B}$ satisfies (2.11) and (2.12). As a result, $\{\cdot, P_0\}' = P_0 \{\cdot, P_0\} P_0 = 0$ and $\{P_0, \cdot\}' = P_0 \{P_0, \cdot\} P_0 = 0$. It is easy to check that the following Leibniz rule holds for $\{\cdot, \cdot\}$ in $M_n(\mathcal{A})$:

$$\{A M, N\} = A \{M, N\} + M \{A, N\}, \quad M, N \in M_n(\mathcal{A}), \quad A \in \mathcal{A} \cong \text{center}(M_n(\mathcal{A})).$$

Combining these identities, we get

$$\{\Psi(A_1), \Psi(A_2)\}' = \{A_1 P_0, A_2 P_0\}' = \{A_1, A_2\} P_0 = \Psi(\{A_1, A_2\}),$$

for $A_1, A_2 \in \mathcal{A}$. \square

3 Picard groups acting on deformations

Let \mathcal{A} be a commutative, unital k -algebra.

By Proposition 2.21, an $(\mathcal{A}, \mathcal{A})$ -equivalence bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$ canonically defines an automorphism of the set $\text{Def}(\mathcal{A})$,

$$\Phi_{\mathcal{E}} : \text{Def}(\mathcal{A}) \longrightarrow \text{Def}(\mathcal{A}). \quad (3.1)$$

We observe that the map $\Phi_{\mathcal{E}}$ only depends on the isomorphism class of \mathcal{E} . We will abuse notation and simply write \mathcal{E} to denote its isomorphism class in $\text{Pic}(\mathcal{A})$ or $\text{Pic}_{\mathcal{A}}(\mathcal{A})$ (if $Ax = xA$ for all $x \in \mathcal{E}$, $A \in \mathcal{A}$).

Let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$, $\mathcal{A}' = (\mathcal{A}[[\lambda]], \star')$ be formal deformations of \mathcal{A} .

Lemma 3.1 *The unital $k[[\lambda]]$ -algebras \mathcal{A} and \mathcal{A}' are Morita equivalent if and only if there exists $\mathcal{E} \in \text{Pic}_{\mathcal{A}}(\mathcal{A})$ and $\psi \in \text{Aut}(\mathcal{A})$ with $\Phi_{\mathcal{E}}([\star]) = [\psi^*(\star')]$.*

PROOF: If $\Phi_{\mathcal{E}}([\star]) = [\psi^*(\star')]$, then \mathcal{A} and \mathcal{A}' are Morita equivalent by Propositions 2.14 and 2.21. Conversely, if \mathcal{A} and \mathcal{A}' are Morita equivalent, then there exists a full idempotent $\mathbf{P} \in M_n(\mathcal{A})$ so that $\mathcal{A}' \cong \mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P}$. By Lemma 2.20, $\mathbf{P} = P_0 + O(\lambda)$ with P_0 full. We know that (see (2.6)) $\mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P}$ is isomorphic to $P_0 M_n(\mathcal{A}) P_0 [[\lambda]]$ as a $k[[\lambda]]$ -module, and since it is also isomorphic to $\mathcal{A}[[\lambda]]$, we must have $P_0 M_n(\mathcal{A}) P_0 \cong \mathcal{A}$. As in Remark 2.10, $\mathcal{E} = P_0 \mathcal{A}^n$ is an $(\mathcal{A}, \mathcal{A})$ -equivalence bimodule satisfying $xA = Ax$, for all $x \in \mathcal{E}$ and $A \in \mathcal{A}$, and \mathcal{A}' is isomorphic to the deformations in the class $\Phi_{\mathcal{E}}([\star])$. The result then follows from Proposition 2.14. \square

We recall that the unit element in $\text{Pic}_{\mathcal{A}}(\mathcal{A})$ is given by (the isomorphism class of) ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$.

Lemma 3.2 *If ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, then $\Phi_{\mathcal{E}} = \text{id}$.*

PROOF: Let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of \mathcal{A} . Then \mathcal{A} itself, regarded as a right module over \mathcal{A} , provides a deformation of $\mathcal{E}_{\mathcal{A}}$. Since $\text{End}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$, it follows that $\Phi_{\mathcal{E}}([\star]) = [\star]$. \square

Lemma 3.3 *Let $\mathcal{E}, \mathcal{E}' \in \text{Pic}_{\mathcal{A}}(\mathcal{A})$, and $\mathcal{E}'' = \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}$. Then $\Phi_{\mathcal{E}''} = \Phi_{\mathcal{E}'} \circ \Phi_{\mathcal{E}}$.*

PROOF: Let $\mathcal{A} = (\mathcal{A}[[\lambda]], \star)$, $\mathcal{A}' = (\mathcal{A}[[\lambda]], \star')$, and $\mathcal{A}'' = (\mathcal{A}[[\lambda]], \star'')$ be formal deformations of \mathcal{A} so that $[\star'] = \Phi_{\mathcal{E}}([\star])$ and $[\star''] = \Phi_{\mathcal{E}'}([\star'])$. Let \mathcal{E} be a deformation of \mathcal{E} corresponding to \star , and \mathcal{E}' be a deformation corresponding to \star' . We know that $\mathcal{A}' \cong \text{End}_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{A}'' \cong \text{End}_{\mathcal{A}'}(\mathcal{E}')$. As discussed in Section 2.1, $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E}$ is an $(\mathcal{A}'', \mathcal{A}')$ -equivalence bimodule, so $\mathcal{A}'' \cong \text{End}_{\mathcal{A}'}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E})$. Since $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E}$ is a f.g.p.m. over \mathcal{A} , it follows (see (2.6)) that it is of the form $V[[\lambda]]$, where $V \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E} / (\lambda \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E})$ as a k -module. But

$$\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E} / (\lambda \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E}) \cong \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}.$$

Hence $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{E}$ is a deformation of $\mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}$, and the conclusion follows. \square

The following lemma is a simple corollary of Theorem 2.25.

Lemma 3.4 *Let $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ and $\star' = \sum_{r=0}^{\infty} C'_r \lambda^r$ be formal deformations of \mathcal{A} such that $[\star'] = \Phi_{\mathcal{E}}([\star])$, for some $\mathcal{E} \in \text{Pic}_{\mathcal{A}}(\mathcal{A})$. Then \star and \star' correspond to the same Poisson bracket, i.e.*

$$C_1(A_1, A_2) - C_1(A_2, A_1) = C'_1(A_1, A_2) - C'_1(A_2, A_1), \quad \text{for all } A_1, A_2 \in \mathcal{A}.$$

As a result, we can state the following theorem.

Theorem 3.5 *Let \mathcal{A} be a commutative, unital k -algebra. Then $\Phi : \text{Pic}_{\mathcal{A}}(\mathcal{A}) \rightarrow \text{Aut}(\text{Def}(\mathcal{A}))$, $\mathcal{E} \mapsto \Phi_{\mathcal{E}}$, defines an action of $\text{Pic}_{\mathcal{A}}(\mathcal{A})$ on the set $\text{Def}(\mathcal{A})$, preserving Poisson brackets. Moreover, two formal deformations of \mathcal{A} , \star and \star' , are Morita equivalent if and only if there exists $\psi \in \text{Aut}(\mathcal{A})$ such that $[\star]$ and $[\psi^*(\star')]$ lie in the same orbit of Φ .*

4 Semiclassical geometry of quantum line bundles

Henceforth $\mathcal{A} = C^{\infty}(M)$, the algebra of complex-valued smooth functions on a manifold M , and we will restrict ourselves to differential deformations of \mathcal{A} (i.e., star products). As we noted in Example 2.11, $\text{Pic}(M) \cong H^2(M, \mathbb{Z})$ (group of isomorphism classes of complex line bundles over M) can be naturally identified with $\text{Pic}_{\mathcal{A}}(\mathcal{A})$ through the map $\text{Pic}(M) \ni L \mapsto \mathcal{E} = \Gamma^{\infty}(L) \in \text{Pic}_{\mathcal{A}}(\mathcal{A})$.

The following result is a consequence of (2.10) and Theorem 3.5.

Theorem 4.1 *Let (M, π) be a Poisson manifold. There is a canonical action*

$$\Phi : \text{Pic}(M) \times \text{Def}_{\text{diff}}(M, \pi) \longrightarrow \text{Def}_{\text{diff}}(M, \pi),$$

and two star products \star and \star' on (M, π) are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\star]$ and $[\psi^(\star')]$ lie in the same orbit of Φ .*

The goal of the remainder of this work is to understand the action Φ and the orbit space $\text{Def}_{\text{diff}}(M, \pi)/\text{Pic}(M)$. To this end, we will study the semiclassical geometry of deformed line bundles over Poisson manifolds. A comparison between the objects arising as ‘‘first-order’’ approximations to deformed vector bundles and the usual notion of Poisson module [29] is discussed in [4].

Let (M, π) be a Poisson manifold, and let $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ be a star product on M satisfying

$$C_1(f, g) - C_1(g, f) = \pi(df, dg), \quad f, g \in C^{\infty}(M).$$

Let $L \rightarrow M$ be a complex line bundle over M , and let $\mathcal{E} = \Gamma^\infty(L)$. Let us fix a deformation quantization of L with respect to \star , $\mathcal{E} = (\mathcal{E}[[\lambda]], \bullet)$, and pick $\star' \in \Phi_{\mathcal{E}}([\star])$. Since $\mathcal{A}' = (C^\infty(M)[[\lambda]], \star') \cong \text{End}_{\mathcal{A}}(\mathcal{E})$, there is a natural left action of \mathcal{A}' on \mathcal{E} that can be written

$$f \bullet' s = fs + \sum_{r=1}^{\infty} R'_r(f, s),$$

for bidifferential maps $R'_r : C^\infty(M) \times \mathcal{E} \rightarrow \mathcal{E}$. It is clear that \bullet' , \bullet make $\mathcal{E}[[\lambda]]$ into an $(\mathcal{A}', \mathcal{A})$ -bimodule.

Definition 4.2 *Let $L \rightarrow M$ be a complex line bundle over a Poisson manifold M . Fix $\star = \sum_r C_r \lambda^r$ on M , and $\star' = \sum_r C'_r \lambda^r \in \Phi_{\mathcal{E}}([\star])$. A triple $(\mathcal{L}[[\lambda]], \bullet, \bullet')$ is called a bimodule quantization of L corresponding to \star, \star' .*

The following equations relate \star, \star', \bullet and \bullet' .

$$(f \star' g) \bullet s = f \bullet' (g \bullet' s), \quad (4.1)$$

$$s \bullet (f \star g) = (s \bullet f) \bullet g, \quad (4.2)$$

$$(f \bullet' s) \bullet g = f \bullet' (s \bullet g). \quad (4.3)$$

Let $R : \mathcal{E} \times C^\infty(M) \rightarrow \mathcal{E}$ be defined by

$$R(s, f) := R_1(s, f) - R'_1(f, s). \quad (4.4)$$

Since $[\star]$ and $\Phi_{\mathcal{E}}([\star])$ correspond to the Poisson bracket $\{\cdot, \cdot\}$ on M (by Lemma 3.4), we may assume $C_1 = C'_1$ in Definition 4.2.

Proposition 4.3 *The map R is a contravariant connection on L .*

PROOF: We must show that R satisfies (A.4), (A.5) in Appendix A. Note that (4.2) yields, in first order,

$$R'_1(fg, s) + C'_1(f, g)s = R'_1(f, gs) + fR'_1(g, s). \quad (4.5)$$

Similarly, (4.3) yields

$$R_1(s, fg) + sC_1(f, g) = R_1(sf, g) + R_1(s, g)f. \quad (4.6)$$

We finally note that (4.3) implies that

$$R_1(fs, g) + R'_1(f, s)g = R'_1(f, sg) + fR_1(s, g). \quad (4.7)$$

The difference of equations (4.5) and (4.6) yields

$$R(fg, s) = R_1(sf, g) + R_1(s, f)g - R'_1(f, gs) - fR'_1(g, s).$$

But, by (4.7), $R_1(sf, g) = R'_1(f, sg) + fR_1(s, g) - R'_1(f, s)g$. This implies that

$$R(s, fg) = fR(s, g) + gR(s, f),$$

proving that (A.4) is satisfied. Now, switching f and g in (4.5), and subtracting it from (4.6) (assuming $C_1 = C'_1$), we get

$$R(sf, g) = \{f, g\}s + R(s, fg) - gR(s, f) = \{f, g\}s + fR(s, g),$$

proving (A.5). \square

We observe that given \star on M , the contravariant connection R on L depends on the choice of \star' , \bullet and \bullet' . As an example, let us compute it in a concrete situation.

Example 4.4 Fix n , and let $t(\mathbb{C}^n) = M \times \mathbb{C}^n \rightarrow M$ be a trivial bundle. Let $P_0 \in M_n(C^\infty(M))$ be a rank-one idempotent so that $L = P_0t(\mathbb{C}^n)$ is a line bundle over M . For a fixed star product \star on M , we pick an idempotent $\mathbf{P} = P_0 + O(\lambda) \in M_n(\mathcal{A})$. Using J in (2.6), and I in (2.7) to establish $\mathbb{C}[[\lambda]]$ -module isomorphisms $P_0\mathcal{A}^n[[\lambda]] \rightarrow \mathbf{P} \star \mathcal{A}^n$, and $P_0M_n(\mathcal{A})P_0[[\lambda]] \rightarrow \mathbf{P} \star M_n(\mathcal{A}) \star \mathbf{P}$, respectively, an explicit computation (in the spirit of Proposition 2.24) shows that $R_1(s, f) = P_0C_1(s, f)$ and $R'_1(f, s) = P_0C_1(f, s)$, where $C_1(f, s)_i = C_1(s_i, f)$ and $C_1(s, f)_i = C_1(f, s_i)$, $i = 1, \dots, n$. Thus

$$R(s, f) = P_0\{s, f\} = \nabla_{X_f} s,$$

where $\nabla = P_0d$ is the adapted connection on L , and X_f the hamiltonian vector field of f .

Let D be a contravariant connection on L induced by an ordinary connection ∇ (i.e., $D_{df}s = \nabla_{X_f}s$). Fix \star on M . It follows from Example 4.4 and [28, Thm. 1.1] that we can choose a bimodule quantization of L so that $R = D$. This in fact holds for any contravariant connection (not necessarily induced by an ordinary one), as discussed in [4].

5 The semiclassical limit of Morita equivalent star products

5.1 Semiclassical curvature

Let (M, π) be a Poisson manifold, and suppose $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ and $\star' = \sum_{r=0}^{\infty} C'_r \lambda^r$ are star products on M , with $C_1 = C'_1 = \frac{1}{2}\{\cdot, \cdot\}$. We can associate a Poisson cohomology class to the pair $[\star], [\star']$, measuring the obstruction for these star products being equivalent modulo λ^3 [3, Prop. 3.1].

Lemma 5.1 Suppose \star and \star' are star products with $C_1 = C'_1 = \frac{1}{2}\{\cdot, \cdot\}$. The map

$$(df, dg) \mapsto (C_2 - C'_2)(f, g) - (C_2 - C'_2)(g, f)$$

defines a d_π -closed bivector field $\tau \in \Gamma^\infty(\wedge^2 TM)$. Moreover, the class $[\tau]_\pi \in H_\pi^2(M)$ depends only on the equivalence classes of \star and \star' .

PROOF: The fact that τ is a closed bivector field was proven in [3, Prop. 3.1].

Suppose $\hat{\star}$ is a star product equivalent to \star :

$$f \hat{\star} g = \sum_{r=0}^{\infty} \hat{C}_r \lambda^r = \mathbf{T}^{-1}(\mathbf{T}(f) \star \mathbf{T}(g)),$$

where $\mathbf{T} = \text{id} + \sum_{r=1}^{\infty} T_r \lambda^r$ is an equivalence transformation. Assume $\hat{C}_1 = C_1 = \frac{1}{2}\{\cdot, \cdot\}$. We must show that, if $\hat{\tau}$ is the closed bivector given by the skew symmetric part of $(\hat{C}_2 - C'_2)$, then $[\hat{\tau}]_{\pi} = [\tau]_{\pi}$. The condition $\hat{C}_1 = C_1$ implies that $T_1 \in \text{Der}(C^{\infty}(M))$. Hence $T_1 = \mathcal{L}_X$, for some vector field $X \in \chi(M)$. A simple computation just using the definitions shows that

$$\hat{\tau} = \tau - d_{\pi} X,$$

and the result follows. \square

If M is a symplectic manifold, then the bivector τ defines a closed 2-form $\tilde{\tau}$ by

$$\tilde{\tau}(X_f, X_g) = \tau(df, dg), \quad (5.1)$$

where X_f and X_g are the hamiltonian vector fields of f and g , respectively. The deRham class $[\tilde{\tau}]$ is the one corresponding to $[\tau]_{\pi}$ under the natural isomorphism between de Rham and Poisson cohomologies (see (A.3)). We can state

Lemma 5.2 *Let M be a symplectic manifold, and let \star, \star' be star products with $C_1 = C'_1 = \frac{1}{2}\{\cdot, \cdot\}$. Then $[\tilde{\tau}] \in H_{dR}^2(M)$ depends only on $[\star], [\star']$.*

Suppose now that \star and \star' satisfy $\star' \in \Phi_{\mathcal{E}}([\star])$, where $\mathcal{E} = \Gamma^{\infty}(L)$ for a line bundle $L \rightarrow M$. Let $(\mathcal{E}[[\lambda]], \bullet, \bullet')$ be a bimodule quantization of L corresponding to \star, \star' . Let $R = R_1 - R'_1$ be the contravariant connection on L defined by \bullet, \bullet' , and let Θ_R denote its curvature (see Appendix A).

Theorem 5.3 *For $f, g \in C^{\infty}(M)$, $s \in \mathcal{E}$, we have $\tau(f, g)s = \Theta_R(df, dg)s$.*

PROOF: From (4.2) we get, in second order,

$$R'_2(fg, s) + R'_1(C'_1(f, g), s) + C'_2(f, g)s = R'_2(f, gs) + R'_1(f, R'_1(g, s)) + fR'_2(g, s). \quad (5.2)$$

Similarly, from (4.3) we get

$$R_2(s, fg) + R_1(s, C_1(f, g)) + sC_2(f, g) = R_2(sf, g) + R_1(R_1(s, f), g) + R_2(s, f)g. \quad (5.3)$$

Finally, from (4.3) we have

$$R_2(fs, g) + R_1(R'_1(f, s), g) + R'_2(f, s)g = R'_2(f, sg) + R'_1(f, R_1(s, g)) + fR_2(s, g). \quad (5.4)$$

Since we assume that $C_1 = C'_1$, subtracting (5.2) from (5.3) yields

$$\begin{aligned}
R(s, C_1(f, g)) - R(R(s, f), g) + R(R(s, g), f) \\
+ (C_2 - C'_2)(f, g)s &= R'_1(g, R_1(s, f)) - R'_1(R'_1(g, s), f) \\
&\quad + R_2(sf, g) - R'_2(f, gs) \\
&\quad + R_2(s, f)g + R(R_1(s, g), f) \\
&\quad + R(R'_1(s, f), g) - fR'_2(g, s) \\
&\quad + R'_2(fg, s) - R_2(s, fg).
\end{aligned}$$

Using (5.4), we then get

$$\begin{aligned}
R(s, C_1(f, g)) - R(R(s, f), g) + R(R(s, g), f) \\
+ (C_2 - C'_2)(f, g)s &= R(R'_1(f, s), g) + R(R_1(s, g), f) \\
&\quad + R'_2(fg, s) - R_2(s, fg) \\
&\quad + R_2(sf, g) + R_2(gs, f) \\
&\quad - R'_2(f, gs) - R'_2(g, sf).
\end{aligned}$$

Taking the skew-symmetric part of this equation, and recalling that $\{f, g\} = C_1(f, g) - C_1(g, f)$, we finally have

$$\tau(f, g)s = R(s, \{f, g\}) - R(R(s, f), g) + R(R(s, g), f).$$

□

Consider the natural map

$$i : H^2(M, \mathbb{Z}) \longrightarrow H_{dR}^2(M). \quad (5.5)$$

We denote $H_{dR}^2(M, \mathbb{Z}) := i(H^2(M, \mathbb{Z}))$.

Corollary 5.4 *Suppose \star and \star' satisfy $[\star'] \in \Phi_{\mathcal{E}}([\star])$, $\mathcal{E} = \Gamma^\infty(L)$. Then $\frac{i}{2\pi}[\tau]_\pi = c_1^\pi(L) \in H_\pi^2(M, \mathbb{Z})$, where $c_1^\pi(L) = \pi^* c_1(L)$ is the Poisson-Chern class of L . In particular, if M is symplectic, $\frac{i}{2\pi}[\tilde{\tau}] = c_1(L) \in H_{dR}^2(M, \mathbb{Z})$.*

Corollary 5.4 provides an integrality obstruction for Morita equivalent star products on Poisson manifolds. In the next three subsections, we will interpret these results in terms of the characteristic classes of star products.

5.2 The symplectic case

If (M, ω) is a symplectic manifold, the set of equivalence classes of star products on M can be described in terms of the second de Rham cohomology of M [3, 11, 14, 27, 38]: There is a bijection

$$c : \text{Def}_{\text{diff}}(M, \omega) \longrightarrow [\omega] + \lambda H_{dR}^2(M)[[\lambda]]. \quad (5.6)$$

The class $c(\star)$ is called the *characteristic class* of \star .

The following result was proven in [20, Prop. 6.2].

Lemma 5.5 *Let \star, \star' be star products on M , and let $\tilde{\tau}$ be the closed 2-form defined in (5.1). Then $[\tilde{\tau}] = \frac{1}{\lambda}(c(\star') - c(\star)) \bmod \lambda$.*

In order to study the semiclassical limit of $\Phi : \text{Pic}(M) \times \text{Def}_{\text{diff}}(M, \omega) \longrightarrow \text{Def}_{\text{diff}}(M, \omega)$, let $S : H_{dR}^2(M)[[\lambda]] \longrightarrow H_{dR}^2(M)$ be the semiclassical limit map $S(\sum_{r=0}^{\infty} [\omega_r] \lambda^r) = [\omega_1]$. With the identification (5.6), we may consider $S : \text{Def}_{\text{diff}}(M, \omega) \longrightarrow H_{dR}^2(M)$.

Let $L \rightarrow M$ be a complex line bundle, and let $\mathcal{E} = \Gamma^{\infty}(L)$.

Theorem 5.6 *The following diagram commutes:*

$$\begin{array}{ccc} \text{Def}_{\text{diff}}(M, \omega) & \xrightarrow{\Phi_{\mathcal{E}}} & \text{Def}_{\text{diff}}(M, \omega) \\ S \downarrow & & \downarrow S \\ H_{dR}^2(M) & \xrightarrow{\widehat{\Phi}_{\mathcal{E}}} & H_{dR}^2(M), \end{array}$$

where $\widehat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] + \frac{2\pi}{i} c_1(L)$.

PROOF: The proof follows directly from Lemma 5.5 and Corollary 5.4. \square

Recall that $\text{Pic}(M) \cong H^2(M, \mathbb{Z})$ and that the kernel of i (see (5.5)) is given by the torsion elements in $H^2(M, \mathbb{Z})$. We then have the following

Corollary 5.7 *Let (M, ω) be a symplectic manifold, and suppose $H^2(M, \mathbb{Z})$ is free. Then the action $\Phi : H^2(M, \mathbb{Z}) \times \text{Def}_{\text{diff}}(M, \omega) \longrightarrow \text{Def}_{\text{diff}}(M, \omega)$ is faithful.*

Recall that for star products \star, \star' on M , their relative Deligne class is defined by $t(\star, \star') = c(\star) - c(\star') \in \lambda H_{dR}^2(M)[[\lambda]]$. We write $t(\star, \star') = \lambda t_0(\star, \star') + O(\lambda^2)$. We have the following immediate consequence of Theorem 5.6 phrased in terms of relative classes.

Corollary 5.8 *If \star, \star' are Morita equivalent star products on a symplectic manifold (M, ω) , then there exists a symplectomorphism $\psi : M \longrightarrow M$ such that $\frac{i}{2\pi} t_0(\star, \psi^*(\star')) \in H_{dR}^2(M, \mathbb{Z})$. Conversely, for any star product \star on M and $[\alpha] \in H_{dR}^2(M, \mathbb{Z})$, there is a star product \star' Morita equivalent to \star such that $t(\star, \star') = \frac{2\pi}{i} [\alpha] \lambda + O(\lambda^2)$.*

5.3 The Poisson case

For an arbitrary Poisson manifold (M, π) , Kontsevich constructed in [23] a bijection

$$c : \text{Def}_{\text{diff}}(M, \pi) \longrightarrow \{\pi_{\lambda} = \pi + \lambda \pi_1 + \dots \in \chi^2(M)[[\lambda]], [\pi_{\lambda}, \pi_{\lambda}] = 0\}/F, \quad (5.7)$$

where F is the group $\{\exp(\sum_{r=1}^{\infty} D_r \lambda^r), D_r \in \text{Der}(C^{\infty}(M))\}$, acting on formal Poisson structures by $\mathbf{T}(\pi_{\lambda}) = \pi'_{\lambda}$ if and only if $\pi'_{\lambda}(df, dg) = \mathbf{T}^{-1}\pi_{\lambda}(d(\mathbf{T}(f)), d(\mathbf{T}(g)))$, for $\mathbf{T} \in F$. This correspondence is a result of a more general fact [23]: there exists an L_{∞} -quasiisomorphism \mathcal{U} from the graded Lie algebra of multivectors fields on M (with zero differential and Schouten bracket), \mathfrak{g}_1 , into the graded Lie algebra of multidifferential operators on M (with Hochschild differential and Gerstenhaber bracket), \mathfrak{g}_2 . Given such an \mathcal{U} , for every formal Poisson structure π_{λ} we can define a star product $\star_{\pi_{\lambda}}$ by

$$f \star_{\pi_{\lambda}} g := fg + \sum_{r=1}^{\infty} \frac{\lambda^r}{r!} \mathcal{U}_r(\underbrace{\pi_{\lambda} \wedge \dots \wedge \pi_{\lambda}}_r)(f \otimes g), \quad (5.8)$$

where $\mathcal{U}_r : \bigwedge^r \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ are the Taylor coefficients of \mathcal{U} . Moreover, Kontsevich showed that one can choose $\mathcal{U}_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ to be the natural embedding of multivector fields into multidifferential operators (note that this embedding does *not* preserve brackets).

If $\pi_{\lambda} = \pi + \lambda\pi_1 + \dots$ is a formal Poisson structure on M , the integrability equation $[\pi_{\lambda}, \pi_{\lambda}] = 0$ immediately implies that $d_{\pi}\pi_1 = 0$.

Lemma 5.9 *If $\pi_{\lambda} = \pi + \lambda\pi_1 + \dots$ and $\pi'_{\lambda} = \pi + \lambda\pi'_1 + \dots$ are equivalent formal Poisson structures, then $[\pi_1]_{\pi} = [\pi'_1]_{\pi}$.*

PROOF: Let $\mathbf{T} = \exp(\sum_{r=1}^{\infty} D_r \lambda^r) \in F$. A simple computation shows that if $\mathbf{T}(\pi_{\lambda}) = \pi'_{\lambda}$, then

$$\pi'_1 = \pi_1 - d_{\pi}X_1,$$

where $X_1 \in \chi(M)$ is defined by $\mathcal{L}_{X_1} = D_1$. Thus $[\pi_1]_{\pi} = [\pi'_1]_{\pi}$. □

With the identification given in (5.7), we define the semiclassical limit map

$$S : \text{Def}_{\text{diff}}(M, \pi) \longrightarrow H_{\pi}^2(M), \quad S([\pi_{\lambda}]) = [\pi_1]_{\pi},$$

where $[\pi_{\lambda}]$ is the equivalence class of the formal Poisson structure $\pi_{\lambda} = \pi + \lambda\pi_1 + \dots$.

Lemma 5.10 *Let \star and \star' be star products on (M, π) , with $c(\star) = [\pi + \lambda\pi_1 + \dots]$ and $c(\star') = [\pi + \lambda\pi'_1 + \dots]$. Let τ be as in Lemma 5.1. Then $[\tau]_{\pi} = [\pi_1]_{\pi} - [\pi'_1]_{\pi}$.*

PROOF: Since in our convention $C_1^{\text{skew}} = \frac{1}{2}\{\cdot, \cdot\}$, we use Kontsevich's explicit construction for the formal Poisson structure $\frac{1}{2}\pi_{\lambda}$. The expression of Kontsevich's star products in terms of the maps \mathcal{U}_r is

$$\begin{aligned} f \star_{\pi_{\lambda}} g &= fg + \lambda\mathcal{U}_1\left(\frac{1}{2}\pi_{\lambda}\right)(f \otimes g) + \frac{\lambda^2}{2}\mathcal{U}_2\left(\frac{1}{2}\pi_{\lambda} \wedge \frac{1}{2}\pi_{\lambda}\right)(f \otimes g) + \dots \\ &= fg + \frac{\lambda}{2}\pi(df, dg) + \lambda^2\left(\frac{1}{2}\pi_1(df, dg) + \frac{1}{8}\mathcal{U}_2(\pi \wedge \pi)(f \otimes g)\right) + \dots \end{aligned}$$

Since \star is equivalent to \star_{π_λ} , and \star' is equivalent to $\star_{\pi'_\lambda}$, by Lemma 5.10 it suffices to compute τ for \star_{π_λ} and $\star_{\pi'_\lambda}$. It is clear from the expression just above that $\tau = \pi_1 - \pi'_1$. \square

Let $L \rightarrow M$ be a complex line bundle, and $\mathcal{E} = \Gamma^\infty(L)$. The following result follows from Lemma 5.10 and Theorem 5.3.

Theorem 5.11 *The following diagram commutes:*

$$\begin{array}{ccc} \text{Def}_{\text{diff}}(M, \pi) & \xrightarrow{\Phi_{\mathcal{E}}} & \text{Def}_{\text{diff}}(M, \pi) \\ S \downarrow & & \downarrow S \\ H_\pi^2(M) & \xrightarrow{\widehat{\Phi}_{\mathcal{E}}} & H_\pi^2(M), \end{array}$$

where $\widehat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] - \frac{2\pi}{i} c_1^\pi(L) = [\alpha] - \frac{2\pi}{i} \pi^* c_1(L)$.

Hence, for a star product \star on (M, π) , each element in $H_\pi^2(M, \mathbb{Z}) = \pi^* H_{dR}^2(M, \mathbb{Z})$ corresponds to a different equivalence class of star products Morita equivalent to \star . Theorem 5.11 shows, in particular, that the semiclassical limit of Φ is trivial when π induces the trivial map in cohomology; the case $\pi = 0$ will be discussed in Section 5.4.

A bivector field π_1 on a Poisson manifold (M, π) is called an *infinitesimal deformation* of π if $d_\pi \pi_1 = 0$.

Corollary 5.12 *Suppose π_1 is an infinitesimal deformation that extends to a formal Poisson structure π_λ . Then the same holds for $\pi_1 + \alpha$ if $\frac{i}{2\pi}[\alpha]_\pi \in H_\pi^2(M, \mathbb{Z})$.*

5.4 Deformations of the zero Poisson structure

As mentioned in the Section 5.3, Theorem 5.11 does not provide much information about the orbits of star products corresponding to the null Poisson structure. We will show that the picture, in this case, is analogous to Sections 5.2, 5.3, but in higher orders of λ .

Let (M, π) be a Poisson manifold, with $\pi = 0$. For simplicity, we will identify equivalence classes of star products on M with their characteristic classes as in (5.7). In order to understand the action of Φ on $\text{Def}_{\text{diff}}(M, \pi)$, consider the disjoint union

$$\text{Def}_{\text{diff}}(M, \pi) = \bigcup_{m \geq 1} \text{Def}_{\text{diff}}^m(M, \pi) \cup [0], \quad (5.9)$$

where $[0]$ denotes the equivalence class of the trivial formal Poisson structure on M , and $\text{Def}_{\text{diff}}^m(M, \pi)$ is the set of equivalence classes of formal Poisson structures of the form $\pi_\lambda^m = \lambda^m(\pi_m + \lambda\pi_{m+1} + O(\lambda^2))$, $\pi_m \neq 0$, $m \geq 1$. Note that we can decompose each $\text{Def}_{\text{diff}}^m(M, \pi)$ into a disjoint union of sets $\text{Def}_{\text{diff}}^m(M, \pi_m)$, given by equivalence classes of formal Poisson structures of the form $\lambda^m(\pi_m + O(\lambda))$ for a fixed Poisson structure $\pi_m \neq 0$. We can always choose a star product $\star = \sum_{r=0}^\infty C_r \lambda^r$ corresponding to a class in $\text{Def}_{\text{diff}}^m(M, \pi_m)$ with

$C_1 = C_2 = \dots = C_m = 0$. It is easy to check that all the results in the previous subsections of Section 5 hold for such star products, with a shift in order by λ^m . For instance, the same arguments as in Theorem 4.1 show that $\text{Def}_{\text{diff}}^m(M, \pi_m)$ is invariant under Φ .

Corollary 5.13 *The trivial class $[0]$ is a fixed point for Φ .*

Let $S_m : \text{Def}_{\text{diff}}^m(M, \pi_m) \longrightarrow H_{\pi_m}^2(M)$ be defined by $S_m(\lambda^m(\pi_m + \lambda\pi_{m+1} + O(\lambda^2))) = [\pi_{m+1}]_{\pi_m}$. Let $L \rightarrow M$ be a line bundle and $\mathcal{E} = \Gamma^\infty(L)$. Just as in Theorem 5.11, one can show the following theorem.

Theorem 5.14 *The following diagram commutes:*

$$\begin{array}{ccc} \text{Def}_{\text{diff}}^m(M, \pi_m) & \xrightarrow{\Phi_{\mathcal{E}}} & \text{Def}_{\text{diff}}^m(M, \pi_m) \\ S_m \downarrow & & \downarrow S_m \\ H_{\pi_m}^2(M) & \xrightarrow{\hat{\Phi}_{\mathcal{E}}} & H_{\pi_m}^2(M), \end{array}$$

where $\hat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] - \frac{2\pi}{i}(\pi_m^* c_1(L))$.

Hence, for a star product in $\text{Def}_{\text{diff}}^m(M, \pi_m)$, each element in $H_{\pi_m}^2(M, \mathbb{Z})$ corresponds to an equivalence class of star products Morita equivalent to it.

A Poisson cohomology, contravariant connections and Poisson-Chern classes

Let (M, π) be a Poisson manifold. The Poisson tensor π defines a bundle morphism

$$\tilde{\pi} : T^*M \longrightarrow TM, \quad \alpha \mapsto \pi(\cdot, \alpha),$$

inducing a map on sections $\tilde{\pi} : \Omega^1(M) \longrightarrow \chi(M)$. The vector field $\tilde{\pi}(df) = X_f$ is called the *hamiltonian vector field* of f . We can use $\tilde{\pi}$ to define a Lie algebra bracket on $\Omega^1(M)$:

$$[\alpha, \beta] = -\mathcal{L}_{\tilde{\pi}(\alpha)}\beta + \mathcal{L}_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta)), \quad \alpha, \beta \in \Omega^1(M). \quad (\text{A.1})$$

The map $-\tilde{\pi} : \Omega^1(M) \longrightarrow \chi(M)$ is a Lie algebra homomorphism, and this makes T^*M into a *Lie algebroid* (see [9, Chp. 16]).

The Poisson tensor $\pi \in \chi^2(M)$ can be used to define a differential

$$d_\pi : \chi^k(M) \longrightarrow \chi^{k+1}(M), \quad d_\pi = [\pi, \cdot], \quad (\text{A.2})$$

where $[\cdot, \cdot]$ is the Schouten bracket [35].

Definition A.1 *The cohomology groups of the complex (χ^\bullet, d_π) are called the Poisson cohomology groups of M and denoted $H_\pi^k(M)$.*

The map $\tilde{\pi}$ induces a map $\pi^* : \Omega^{\bullet(M)} \longrightarrow \chi^\bullet(M)$ intertwining differentials, and therefore gives rise to a morphism in cohomology

$$\pi^* : H_{dR}^k(M) \longrightarrow H_\pi^k(M), \quad (\text{A.3})$$

which is an isomorphism when π is symplectic. We define integral (resp. real) Poisson cohomology as the image of integral (resp. real) deRham cohomology classes on M under π^* , i.e., $H_\pi^k(M, \mathbb{Z}) = \pi^* H_{dR}^k(M, \mathbb{Z})$ (resp. $H_\pi^k(M, \mathbb{R}) = \pi^* H_{dR}^k(M, \mathbb{R})$).

The key ingredient in defining contravariant connections on vector bundles over Poisson manifolds is to think of T^*M as a “new” tangent bundle to M , using its Lie algebroid structure (see [16]).

Let $E \rightarrow M$ be a complex vector bundle over a Poisson manifold (M, π) .

Definition A.2 A contravariant connection on E is a \mathbb{C} -linear map $D : \Gamma^\infty(E) \otimes \Omega^1(M) \longrightarrow \Gamma^\infty(E)$ so that

$$i.) \ D_{f\alpha} s = f D_\alpha s$$

$$ii.) \ D_\alpha(fs) = f D_\alpha s + \alpha(X_f)s,$$

for $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$. The curvature of a contravariant connection D is the map $\Theta_D : \Omega^1(M) \otimes \Omega^1(M) \longrightarrow \text{End}(\Gamma^\infty(E))$,

$$\Theta_D(\alpha, \beta)s = D_\alpha D_\beta s - D_\beta D_\alpha s + D_{[\alpha, \beta]}s.$$

It is easy to see that, if ∇ is any connection (in the usual sense) on E , then it induces a contravariant connection by $D_{df} = \nabla_{X_f}$. On symplectic manifolds this is the only way that contravariant connections can arise. Thus this notion is mostly important in degenerate situations.

A bilinear map $D' : \Gamma^\infty(E) \times C^\infty(M) \longrightarrow \Gamma^\infty(E)$, satisfying

$$D'(s, f \cdot g) = D'(s, f)g + D'(s, g)f, \quad (\text{A.4})$$

$$D'(s \cdot f, g) = D'(s, g)f + s\{f, g\}, \quad (\text{A.5})$$

provides an equivalent definition of a contravariant connection. The definitions are related by the formula

$$D'(s, f) = D_{df}s.$$

If $E = L \rightarrow M$ is a line bundle, then the curvature Θ_D of a contravariant connection defines a bivector field on M , closed with respect to d_π [34]. As in the case of usual connections, its Poisson cohomology is a well-defined class, independent of the connection.

Definition A.3 let D be a contravariant connection on a line bundle $L \rightarrow M$, and let Θ_D be its curvature. We call the class $c_1^\pi(L) := \frac{i}{2\pi}[\Theta_D]_\pi \in H_\pi^2(M)$ the Poisson-Chern class of L .

It is clear that $c_1^\pi(L) = \pi^*(c_1(L))$.

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